MATH 2020 Advanced Calculus II

Tutorial 6
Oct 15,17

1. Find the area of the region bounded by $y = x, y = 2x, xy = 1$ and $xy = 2$.

Solution. Consider the transformation

$$\begin{cases} u &= y/x \\ v &= xy \end{cases}.$$

Then R is mapped bijectively onto the square R' defined by $\{1 \leqslant u, v \leqslant 2\}$. We have

$$\begin{cases} x &= \sqrt{v/u} \\ y &= \sqrt{uv} \end{cases}$$

and

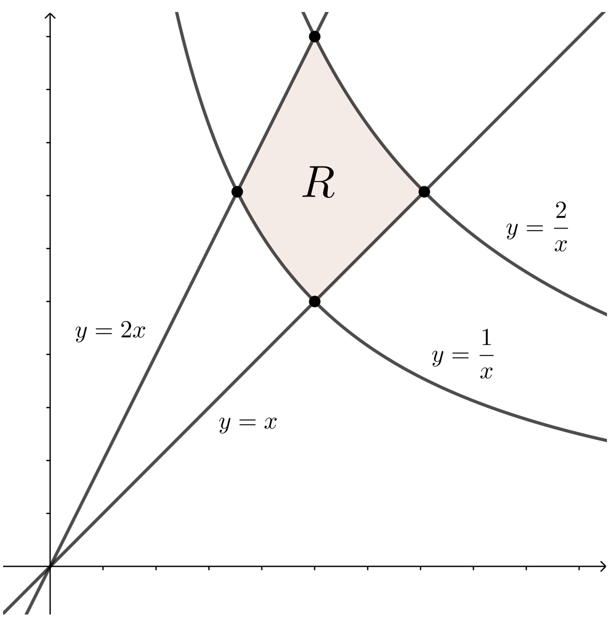
$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{bmatrix} -\frac{1}{2}\sqrt{\frac{v}{u^3}} & \frac{1}{2}\frac{1}{\sqrt{uv}} \\ \frac{1}{2}\sqrt{\frac{v}{u}} & \frac{1}{2}\sqrt{\frac{u}{v}} \end{bmatrix}$$

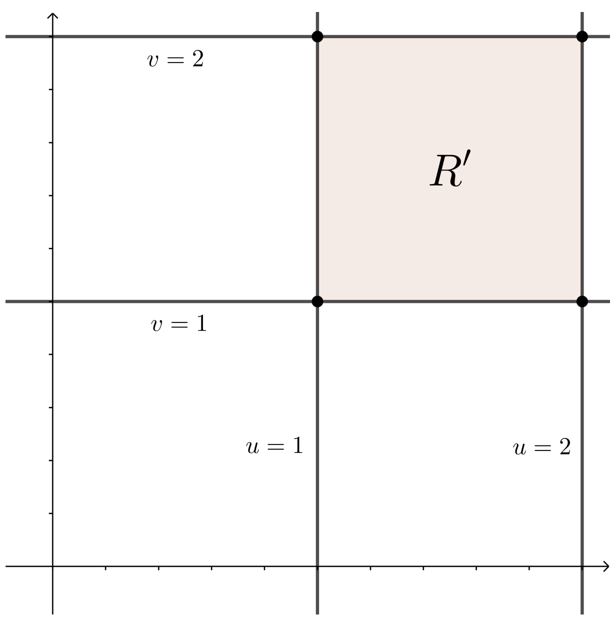
So

$$\left|\left|\frac{\partial(x,y)}{\partial(u,v)}\right|\right| = \frac{1}{4}\left|-\frac{1}{u} - \frac{1}{u}\right| = \frac{1}{2u}.$$

It follows that

$$\begin{aligned} \text{Area} &= \iint_R dydx \\ &= \iint_{R'} \left|\left|\frac{\partial(x,y)}{\partial(u,v)}\right|\right| dvdu \\ &= \int_1^2 \int_1^2 \frac{1}{2u} dvdu \\ &= \frac{1}{2}\ln 2. \end{aligned}$$





2. (a) Evaluate $I_1 = \int_0^\infty e^{-xy}\sin y dy$ and $I_2 = \int_0^\infty e^{-xy}\cos y dy$ (where x is regarded as constant).

(b) Show that $\int_0^\infty \frac{x\sin x}{x^2+1}dx = \int_0^\infty \frac{\cos x}{x^2+1}dx$.

Solution.

(a) By integration by parts,

$$\begin{aligned} I_1 &= \left[e^{-xy}(-\cos y)\right]_0^\infty - x\int_0^\infty e^{-xy}\cos y dy \\ &= 1 - xI_2 \\ I_2 &= \left[e^{-xy}(\sin y)\right]_0^\infty + x\int_0^\infty e^{-xy}\sin y dy \\ &= xI_1. \end{aligned}$$

So we have

$$\begin{cases} I_1 &= 1 - xI_2 \\ I_2 &= xI_1 \end{cases} \implies \quad I_1 = \frac{1}{x^2+1} \quad \text{and} \quad I_2 = \frac{x}{x^2+1}.$$

(b)

$$\begin{aligned} \int_0^\infty \frac{x\sin x}{x^2+1}dx &= \int_0^\infty\int_0^\infty e^{-xy}\sin x\cos y dy dx \\ &= \int_0^\infty\int_0^\infty e^{-xy}\sin x\cos y dx dy \\ &= \int_0^\infty \frac{\cos y}{y^2+1}dy \\ &= \int_0^\infty \frac{\cos x}{x^2+1}dx \qquad \text{(changing the dummy variables)} \end{aligned}$$

Remark. In fact, it is possible to evaluate these two integrals, say the second one, as follows:

Let $I(a) = \int_0^\infty \frac{\cos ax}{x^2+1}dx$, regarded as a function of a. Then $I(0) = \int_0^\infty \frac{dx}{x^2+1} = \frac{\pi}{2}$ and $I(1)$ is what we are computing. Taking the derivative:

$$\begin{aligned} I'(a) &= -\int_0^\infty \frac{x\sin ax}{x^2+1}dx \\ &= -\int_0^\infty\int_0^\infty e^{-xy}\cos y\sin ax dy dx \qquad \text{(by part (a))} \\ &= -\int_0^\infty\int_0^\infty e^{-xy}\cos y\sin ax dx dy \\ &= -\int_0^\infty \frac{a\cos y}{y^2+a^2}dy \\ &= -\int_0^\infty \frac{\cos at}{t^2+1}dt \\ &= -I(a). \end{aligned}$$

(Here the fourth equality follows from a slight generalization of part (a) and the fifth equality follows from the substitution $y = at$.)

Solving this simple ODE, we have

$$I(a) = I(0)e^{-a} = \frac{\pi}{2}e^{-a},$$

and hence

$$\int_0^\infty \frac{\cos x}{x^2+1}dx = \frac{\pi}{2e}.$$

3. Evaluate $\displaystyle\int_0^\infty \frac{\sin x}{x}dx.$

Solution. Observe that

$$\frac{1}{x} = \int_0^\infty e^{-xy}dy.$$

So

$$\begin{aligned}
\int_0^\infty \frac{\sin x}{x}dx &= \int_0^\infty \int_0^\infty e^{-xy}\sin x dy dx \\
&= \int_0^\infty \int_0^\infty e^{-xy}\sin x dx dy \\
&= \int_0^\infty \frac{dy}{y^2+1} \qquad \text{(by Q2 part (a))} \\
&= \frac{\pi}{2}.
\end{aligned}$$